5. I. I. Novikov, "One case of the motion of an incompressible fluid in a cylindrical channel," Tr. Voenno-Morskoi Akad., Leningrad (1945).
6. M. A. Gol'dshtik, Vortex Flows [in Russian], Nauka, Novosibirsk (1981).
7. Yu. I. Khavkin, Swirl Sprayers [in Russian], Mashinostroenie, Leningrad (1976).

DYNAMICS OF A UNIFORM TURBULENT LAYER IN A STRATIFIED FLUID

V. Yu. Lyapidevskii

UDC 532.526;51.465

At large Reynolds numbers, a flow of an incompressible density-stratified fluid separates into alternating layers of turbulent and laminar flow [1]. Turbulent flow develops under the influence of shear instability or as a result of boundary conditions. One problem encountered in the description of such flows is representing the process of entrainment of surrounding fluid into a turbulent layer in parametric form [2, 3]. The most important factor here is the effect of stratification on the rate of entrainment. Various mechanisms of instability development, leading to mixing [2], become predominant. The specific mechanism that prevails depends on the relation between buoyancy and inertia. When mixing occurs, the rate of entrainment may change by several orders of magnitude. Since the flow region in which a given type of instability will develop is unknown beforehand, it is interesting to attempt to construct a model of stratified flow that will uniquely describe the entrainment process.

One possible approach to the solution of this problem is demonstrated below by using the example of the evolution of a turbulent layer in a quiescent fluid of another density. This class of flows includes submerged jets, gravitational flows, and the movement of the uniform upper layer of an ocean to a lower depth by wind [1]. The model that is constructed should reflect such experimentally observed flow properties as the potential for controlling the entrainment process by altering the conditions downflow, the sharp reduction in entrainment velocity with the transition from supercritical to subcritical flow, and the phenomenon of the excitation of short internal waves at the boundary of the turbulent layer in flows with a velocity shift [2].

Here we examine these phenomena on the basis of equations of motion of the layer which constitute a variant of the equations of "shallow water." Allowance is made for mixing. The equations of motion were derived from conservation laws in a manner similar to [4]. The rate of entrainment of fluid into the turbulent layer is assumed to be proportional to the velocity of "large eddies" that are commensurate with the thickness of the layer [2, 5]. Analysis of traveling waves in the system in question shows that solutions of the solitary-wave or jump-wave types describe the naturally-observed generation of short-period internal waves at the crests of longer (tidal) waves [6, 7].

Equations of "Shallow Water." In the Boussinesq approximation, the equations of a thin horizontal layer of fluid of thickness h and density ρ moving at velocity u in a quiescent fluid of density ρ_r have the form

$$\begin{gathered} h_t + (hu)_x = \sigma q,\ (bh)_t + (bhu)_x = 0, \\ (hu)_t + (hu^2 + 0.5bh^2)_x = 0, \\ (h(u^2 + e + bh))_t + (hu(u^2 + e + 2bh))_x = 0. \end{gathered} \tag{1}$$

Here, t is time; x, horizontal coordinate; $b = (\rho - \rho_r)g/\rho_r$, buoyancy; g, the vertical component of acceleration due to gravity; e is the energy associated with pulsative motion. The rate of entrainment of the quiescent fluid into the uniform layer is assumed to be proportional to the velocity of "large eddies" q characterizing pulsative motions in the layer. The eddies are comparable in size to the main flow [5]. System (1) will be closed if we put

$$e = q^2. \tag{2}$$

The constant σ determines the ratio of the scales of fluctuational and average motions in

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 73-76, March-April, 1989. Original article submitted August 5, 1988.

0021-8944/89/3002-0235$12.50

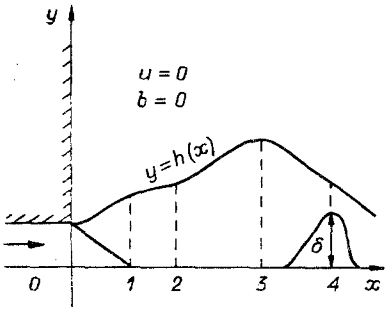

Fig. 1

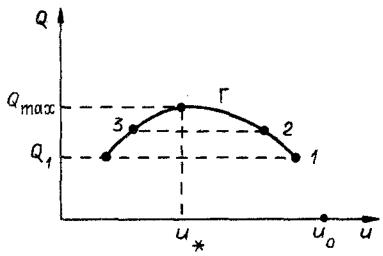

Fig. 2

a flow with a velocity shift. Its numerical value $\sigma = 0.15$ was found in [8] in a study of entrainment in a uniform fluid and is independent of stratification [2].

System (1), (2) leads to the equation

$$(hq)_t + (hqu)_x = 0.5\sigma(u^2 + e - bh). \tag{3}$$

Equation (3) shows that in the case of stable stratification ($b > 0$), system (1), (2) describes not only the entrainment of quiescent fluid into the uniform layer ($q > 0$) but also the wave process at the boundary of the layer. In this process, kinetic energy is transformed to potential energy due to the entrainment of the lighter fluid and conversely (the velocity q is sign-changing). This leads to the excitation of waves of a certain length in the flow. If the newly-chosen averaging scale is comparable to the length of these waves, then the mean value of velocity $\langle q \rangle$ will differ appreciably from the instantaneous value and, thus $\langle e \rangle > \langle q^2 \rangle$. As a result, after averaging over the new scale, Eqs. (1) and (3) become independent (the averaging signs are dropped) and form a closed system. The differential consequence of this system is the equation of "dissipation" of the energy associated with small-scale motion $e_0 = e - q^2$: $e_{0t} + ue_{0x} = -\sigma q e_0/h$. If $e_0 = 0$ at the initial moment, then in the continuous solution of (1) and (3) $e_0 \equiv 0$ at all moments of time, i.e., systems (1), (2) and (1), (3) are equivalent in this case.

Submerged Jet. The main properties of the steady-state solutions of system (1), (3) can be illustrated by using the example of the problem of a submerged jet (Fig. 1). The initial section 0-1 corresponds to the mixing layer, which is described by the equations of two-layer "shallow water" [2] and is not examined here. At a sufficiently large initial Froude number $Fr_0 = u_0/(b_0h_0)^{1/2}$, the mixing layer reaches the bottom and changes downflow into a submerged jet. The flow is supercritical on section 1-2 ($Fr = u/(bh)^{1/2} > 1$) and intensive entrainment of the quiescent fluid occurs. One feature of the mixing fluids is the possibility of control of the internal hydraulic jump 2-3 effecting the transition from subcritical flow ($Fr < 1$). Control could be exercised by means of control section 4 [9].

With an obstacle of increasing height δ, the amount of fluid drawn into the jet decreases. Meanwhile, in both the hydraulic jump 2-3 and in the subcritical flow 3-4, entrainment is very slight compared to the section 1-2 of supercritical flow.

To explain the observed features of flow in the jet from the viewpoint of the above model, we will examine possible steady-state solutions of system (1), (3) in the plane (u, Q), where $Q = hu$ is the discharge of the fluid in the jet (Fig. 2). If the state 0 represents uniform potential flow ($h = h_0$, $u = u_0$, $b = b_0$, $e_0 = 0$, $q_0 = 0$), then the mixing layer changes to a submerged jet at $Q > Q_1 = 2Q_0$. The states on the curve Γ at $u < u_* = (b_0h_0u_0)^{1/3}$ correspond to subcritical flow, while the states at $u > u_*$ correspond to supercritical flow. The maximum entrainment regime Q_{max} is realized at the maximum point of the curve Γ, i.e., it corresponds to critical flow ($\delta = 0$). Thus, it is not necessary to use the additional stability condition employed in [10] to find Q_{max}. As a result, the ratios of the experimental values of h_e, u_e, Q_e found in [10] for the maximum entrainment regime to the corresponding values in the present model are very close to unity ($h_e/h_{max} = 0.96$, $u_e/u_{max} = 0.98$, $Q_e/Q_{max} = 0.93$). An increase in the height of the barrier δ is accompanied by the formation of the hydraulic jump 2-3, the relations for which follow from conservation laws (1), (3). In particular, $Q_2 = Q_3$. The dependence of Fr on barrier height beyond the jump was studied in [9] with the assumption that entrainment is absent on section 3-4. The sharp reduction seen in entrainment velocity after the hydraulic jump is explained by the fact that, in contrast to the 0-2 transition (where $q > 0$) the velocity q on 3-4 is sign-changing, i.e., the mean velocity ⟨q⟩ is close to zero due to intensive wave motion inside the layer. An equilibrium or quasiequilibrium state is reached if

$$\langle q\rangle \equiv 0,\ bh \equiv u^2 + e. \tag{4}$$

It is not hard to see that condition (4) can be realized only in subcritical flow $bh < u^2$.

Traveling Waves of System (1), (3). Let a wave propagate at the velocity D against an equilibrium background ($u_0 = 0$, $e_0 = b_0h_0$, $q_0 = 0$), i.e., the sought quantities in (1), (3) depend only on the variable $\xi = x - Dt$ ($D > 0$). Here, system (1), (3) reduces to a system of ordinary differential equations whose solution can be found in quadratures. In dimensionless variables $\eta = h/h_0$, $v = u/D$, $Ri_D = b_0h_0/D^2$, the dependence of ξ on v has the form

$$\xi = \int_{v_1}^{v} \frac{dQ}{\sigma r},\quad r^2 = \frac{1}{2Q^2}\int_{v_1}^{v} g(s)\,dQ^2(s),\quad g(v) = v\left(\frac{3\,Ri_D}{(1-v)^2} - 4\right),\quad Q = (v-1)\eta,$$

$$v_1 = \begin{cases} 0, & Ri_D > 1, \\ 0.5\,Ri_D(\eta_1^2 - 1), & Ri_D \leqslant 1, \end{cases}\quad \eta_1 = \frac{\sqrt{Ri_D^2 + 8\,Ri_D} - Ri_D}{2\,Ri_D}.$$

The following regimes are possible for system (1), (3), depending on the value of the parameter Ri_D:

a) $1 \le Ri_D < 4/3$ (solitary wave),

b) $1/3 < Ri_D < 1$ (jump-wave),

c) $Ri_D = 1/3$ (bore).

At $Ri_D = 4/3$, the velocity of the wave D coincides with the rate of propagation of perturbations in the equilibrium model consisting of the laws of conservation of mass, momentum, and energy [the last three equations in (1)] and condition (4). At $Ri_D = 1$, the wave moves at the velocity of perturbations in system (1), (3). Thus, the continuous solution (solitary wave) is possible within the range of wave velocities D between the equilibrium and frozen rates of disturbance propagation. If $D > \sqrt{b_0h_0}$, then the solution consists of a hydraulic jump, which is adjoined by the periodic solution (jump-wave).

An increase in D is accompanied by a decrease in wave amplitude beyond the jump, while at $D = \sqrt{3b_0h_0}$ the initial equilibrium state changes by means of a jump to a new equilibrium state (bore). The maximum amplitude of the waves after the jump $\sim 0.2h_0$, and the characteristic wavelengths are from $40h_0$ to $60h_0$. An interesting feature of the traveling waves is the presence of two maxima on the wave crest at $Ri_D \sim 1$.

The generation of short-period internal waves at the crest of longer (tidal, standing) waves has been observed by several investigators [6, 7]. In the case when a thermocline is located nearly on the bottom, its abrupt rise accompanies a train of intensive short waves whose parameters (length, amplitude) correspond to the traveling waves of system (1), (3). Thus, some of the features observed in the generation and propagation of short-period internal waves in the neighborhood of a thermocline (separate frequency of oscillation, alternation of wave trains associated with the passage of the crest of a longer wave) can be explained on the basis of the properties of the solution of system (1), (3).

Note. System (1), (3) contains the unique parameter σ. Its numerical value does not affect the structure of the solutions, since it can be excluded from the system by expanding the independent variables. The choice of the numerical value $\sigma = 0.15$ for uniform and stratified fluids in a way represents the hypothesis of "saturation" of the spectrum of motions of the sub-grid scale in fluid flows with a velocity shift.

LITERATURE CITED

1. J. S. Turner, Buoyancy Effects in Fluids, Cambridge Univ. Press, New York (1973).
2. L. V. Ovsyannikov, N. I. Makarenko, V. I. Nalimov, et al., Nonlinear Problems of the Theory of Surface and Internal Waves [in Russian], Nauka, Novosibirsk (1985).
3. O. M. Phillips, "Entrainment," in: Modeling and Predicting the Upper Layers of the Ocean [Russian translation], Gidrometeoizdat, Leningrad (1979).
4. V. Yu. Lyapidevskii, "Modeling two-phase flows on the basis of conservation laws," Din. Sploshnoi Sredy, 76 (1986).

5. V. Yu. Lyapidevskii, Problem of the mixing layer in a uniform fluid," Din. Sploshnoi Sredy, 74 (1986).
6. K. D. Sabinin and V. A. Shulepov, "Toward a model of the frequency spectrum of internal waves in the ocean," Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana, 17, No. 1 (1981).
7. V. A. Ivanov, K. V. Konyaev, and A. N. Serebryanyi, "Groups of intensive internal waves in the shelf region of the ocean," Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana, 17, No. 12 (1981).
8. A. A. Townsend, Structure of Turbulent Shear Flow, Cambridge Univ. Press, New York (1975).
9. D. L. Wilkinson and J. R. Wood, "A rapidly varied flow phenomenon in a two-layer flow," J. Fluid Mech., 47, No. 2 (1971).
10. V. H. Chu and R. E. Baddour, "Turbulent gravity-stratified shear flows," J. Fluid Mech., 138, 353 (1984).

SUPERCRITICAL FLOWS FROM BENEATH A SHIELD

V. I. Nalimov UDC 532.59

The theory of motion of an ideal incompressible heavy liquid with free boundaries is a unique branch of classical hydrodynamics. Interest in such flows exists first, because of their great practical importance and, second, because of the richness, uniqueness, and difficulty in mathematical description of the problems which develop. Many studies have been published concerning precise solutions of the steady-state equations of motion of a vortex-free liquid with free boundaries. Proof of the existence of traveling waves was first offered by Nekrasov in 1921, then independently by Levi-Civita in 1925 for an infinitely deep liquid. Later, Struik, followed by Nekrasov, established analogous theorems for liquids of finite depth. It was assumed in their studies that the flow was subcritical, i.e., that the velocity of the main flow was less than the propagation velocity of infinitely low amplitude waves. In the 1950s a number of studies appeared on steady-state supercritical flows. Works by Zherbe, Moiseev, and Ter-Krikorov proved the existence of supercritical waves above a rough periodic bottom. Information on these and other studies on the same theme can be found in [1, 2]. In 1982, the existence of subcritical flows about a rough aperiodic bottom was established [3]. The existence of combined waves was first strictly proven by Lavrent'ev [4] in 1946 using the variation principles of conformal and quasiconformal transform theory which he developed. Another method of proof was proposed in [5]. Both proofs are based on principles of nonlinear shallow wave theory and show that this theory can be used for asymptotic representation of precise solutions of the combined wave problem. If we admit the possibility of contact of the free surface with rigid boundaries, the corresponding nonlinear boundary conditions become much more complicated. The present study will examine the two-dimensional problem of flow of a heavy vortex-free ideal liquid from underneath a planar horizontal lid over a smooth horizontal bottom. The flow is assumed supercritical: $U > \sqrt{gh_0}$, although the characteristic flow velocity U is assumed to differ little from the critical velocity $\sqrt{gh_0}$ (where g is the acceleration of gravity and h_0 is the characteristic liquid thickness). We will find the approximate form of the free surface and present a method for proving the existence of flows that are not uniform. It is thus just this fact which justifies the approximate solution. The problem to be formulated herein differs greatly from that of the combined wave. Nevertheless, the method of study to be presented below has much in common with that proposed by Friedrichs and Heyers [5].

1. Formulation of the Problem. To describe the liquid flow we choose as independent variables [1] the dimensionless velocity potential φ and flow function ψ. Such a choice of variables allows us to operate in a fixed belt between two flow lines ψ = const rather than in a partially unknown flow region. As is well known, the complex velocity potential $\chi = \varphi + i\psi$ is an analytical function of the variable $z = x + iy$. The conjugate complex velocity $\bar{w} = d\chi/dz$ is also an analytical function of z. After the substitution $w = \exp\{-i(\theta + i\tau\}$ the problem of liquid flow is reduced [1] to search for an analytic function $\theta + i\tau$ of

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 77-81, March-April, 1989. Original article submitted August 18, 1988.

